Section 15.3

Two new derivatives

Definition: Partial Derivatives
The **partial derivative of f with respect to x at the point** (a, b) is

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) := \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

The **partial derivative of f with respect to y at the point** (a, b) is

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b) := \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

We may rewrite the above limits to elucidate the methods by which we compute partial derivatives.

Example 15.3.1. Compute the partial derivatives of $f(x, y) = \sin xy$.

Example 15.3.2. Compute the partial derivatives of $g(x, y) = x^2 e^{xy}$.

Higher-Order Partial Derivatives

For a function $z = f(x, y)$, there are 2 partial derivatives, each of which are functions of x and y. Hence there are 2(2) second order partial derivatives; generally, 2^n nth order partial derivatives.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Example 15.3.3. Find the four second order partial derivatives of $f(x, y) = 3x^4 y - 2xy + 5xy^3$.

Theorem 15.4: Clairaut' Theorem
Suppose f is defined on an open set D of $\mathbb{R}^2$ and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Example 15.3.4. Define and compute the partial derivatives for the function $f(x, y, z) = e^{-xy} \cos z$.

On Differentiability

Recall the intrinsic property of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and generalize to $f(x, y)$.

Theorem 15.5: Conditions for Differentiability
Suppose the function f has partial derivatives f_x and f_y defined on an open set containing (a, b) with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b).

Theorem 15.6: Differentiable Implies Continuous
If a function f is differentiable at (a, b), then it is continuous at (a, b).

Example 15.3.5. Determine the differentiability of $f(x, y) = 1 - |xy|$ at $(0, 0)$.